the existence of a stability multiplier of the form $(1 \pm qs)$, $q > 0$ and stability can be proved with Popov-type nonlinearities.

2) Let $\mathscr{R}_0^{\pi/2}$ $(\mathscr{R}_{-\pi/2}^0)$ denote the strip on the $\omega - \varphi$ plane between the lines $\varphi = 0$ and $\varphi = +\pi/2(-\pi/2)$. Only if either $\tilde{\mathscr{R}} \cap \mathscr{R}_0^{\pi/2}$ or $\tilde{\mathscr{R}} \cap \mathscr{R}_{-\pi/2}^0$ is nonnull for all $\omega \in [0, \infty)$, then a stability multiplier of the *RL* or *RC* form[2] is possible and stability of the system with monotone nonlinearities can be inferred by constructing this function.

*Determination of the allowable sector for monotone nonlinearities*

The analysis in the previous section was centered on nonlinearities belonging to the infinite sector. However, for the finite sector problem i.e., where $0 \leqq xf(x) \leqq Kx^2$, $K < \infty$, the region $\tilde{\mathscr{R}}$ has to be constructed from $\Phi(G + 1/K)$ instead of $\Phi(G)$. It must be noted that the region $\tilde{\mathscr{R}}$ constructed from $\Phi(G + 1/K)$ will be larger than that constructed from $\Phi(G)$ and in many a case, a system for which stability cannot be proved with a nonlinearity of a particular class in the infinite sector, can be proved to be stable with a similar nonlinearity in a finite sector. This enlargement in $\tilde{\mathscr{R}}$ as $K$ is reduced can be used to determine the largest sector within which the nonlinearities of a particular class should lie, for the stability of the system. The procedure for determining the allowable sector $[0, K_M]$ for monotone nonlinearities will now be indicated.

Since the systems considered will be stable with all linear feedback, the Hurwitz sector will be $[0, \infty]$. At the other extreme, the Popov sector $[0, K_P]$ can be determined, as shown by Siljak[6] by plotting the envelope of the Popov inequality

$$Re(1 + jq\omega)\, G(j\omega) + 1/K \geqq 0 \;\forall\; \omega \in [0, \infty)$$

on the $1/K - q$ parameter plane.

Now, $K_M$ should satisfy, $K_P \leqq K_M \leqq \infty$. Hence, by constructing the region $\tilde{\mathscr{R}}$ from $\Phi(G + 1/K)$ for different values of $K$ increased in steps from $K = K_P$, the largest value of $K$ for which either $\tilde{\mathscr{R}} \cap \mathscr{R}_0^{\pi/2}$ or $\tilde{\mathscr{R}} \cap \mathscr{R}_{-\pi/2}^0$ (just described) is nonnull for all $\omega \in [0, \infty)$, is determined. Multipliers which have the phase characteristic within this region of intersection are constructed[5] and are analysed to see if they are of the *RL* or *RC* form. If not, a slightly smaller value of $K$ is chosen and the procedure is repeated. Then, $K_M$ is the largest value of $K$ for which these conditions are satisfied.

The determination of $K_M$ will be very useful in such practical problems as the stability analysis of nuclear reactor control systems, wherein $K_M$ fixes the maximum power level for stable operation[7] of the reactor.

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# Average Circumferential Pressure on Inclined Bodies of Revolution at Hypersonic Speed


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## Introduction

A SMALL probe is sometimes fixed to the nose of a missile to permit measurement of the static pressure, and the flight altitude is then inferred from the pressure. Typically, the probe is an axially-symmetric body with several orifices located around the circumference, each connected to a common chamber or manifold where the pressure is sensed. The manifold serves to average the local pressures acting at the orifices so that the measured value is relatively insensitive to angle of attack and roll position. The pressure in the manifold may differ significantly from the average hydrostatic pressure at the probe surface. The extent of the difference depends upon a host of parameters, such as heat transfer and pressure,[1,2] orifice inclination,[3] etc. Further, the pressure along the surface of the probe may depend upon shock-wave boundary-layer interactions which occur when the wave emanating at the juncture of the probe afterbody and the missile nose section coalesces with the probe boundary layer.[4] Finally, the pressure that is to be sensed depends upon the number of orifices and their locations, and that dependence is discussed here for the hypersonic speed regime.

## Derivation of Equation

The pressure on the surface of a body in hypersonic flow as specified by the Newtonian impact theory[5] is $C_p = C_{p,o}\, sin^2 \xi$, where $C_p$ and $C_{p,o}$ are the local and stagnation-point pressure coefficients, respectively, and $\xi$ is the apparent flow incidence angle, the angle between the velocity vector and the plane tangent to the surface at the desired point. If $\delta$ represents the local slope of the surface with respect to the longitudinal axis of a body of revolution and $\alpha$ represents the total angle of attack of the body, the angle of incidence becomes $\xi = \arcsin(\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin\Phi)$ where $\Phi$ is the angular position of the point along the circumference.

When the points are positioned on the circumference in equal increments so that $n(\Delta\varphi) = 2\pi$ rad, where $n$ denotes the total number of points, the incidence angle at any point $j = 1\; 2, \ldots$ may be expressed $\xi = \arcsin\{\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin[\varphi + (j-1)(2\pi/n)]\}$ where $\varphi$ denotes the body aerodunamic roll angle. Therefore, the pressure coefficient at location $j$ becomes

$$C_p = C_{p,o}\{\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin[\varphi + (j-1)(2\pi/n)]\}^2 \qquad (1)$$

Since the coefficient of the numerical average of the local pressures is equal to the average of the local pressure coefficients, the average value is simply

$$\langle C_p \rangle = (1/n) \sum_{j=1}^{n} C_p$$

Hence, the equation describing the average pressure is

$$\langle C_p \rangle = (1/n) C_{p,o} \sum_{j=1}^{n} \{\cos\alpha \sin\delta + \sin\alpha \cos\delta \sin[\varphi + (j-1)(2\pi/n)]\}^2 \qquad (2)$$


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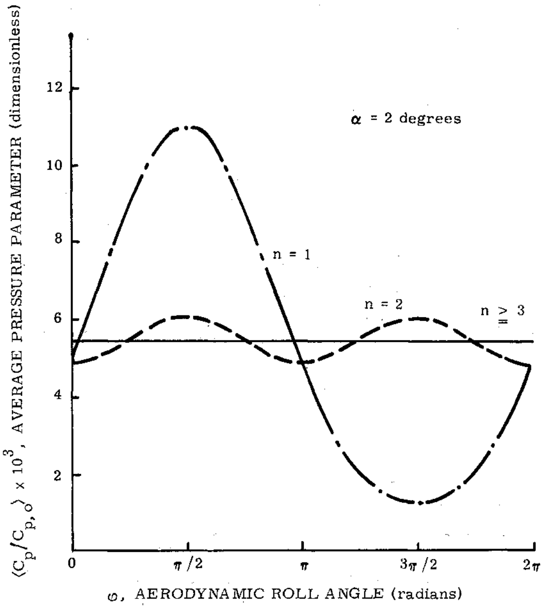


**Fig. 1 Average pressure coefficient at $n$ equally spaced points as a function of roll attitude for 2° angle of attack ($\delta = 4$).**

Since the Newtonian theory does not specify the pressure on surfaces that are situated within a "shadow" zone, Eq. (2) is restricted to angles of attack less than or equal to the local body slope. Further, since the theory does not account for the characteristic overexpansion of flow near the nose of blunt bodies or the decrease of dynamic pressure in the entropy layer that envelops the afterbody, its utility may be confined to configurations with small nose bluntness and small slope.

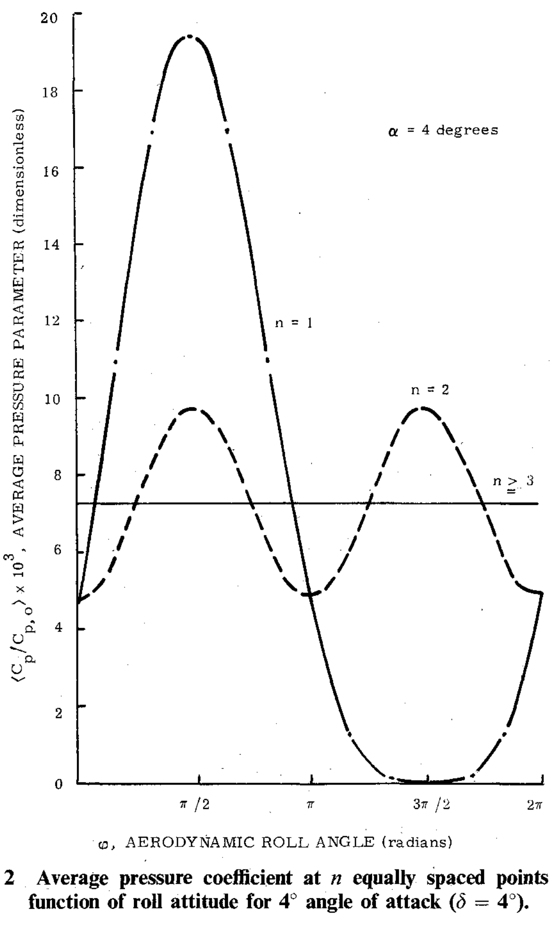


**Fig. 2 Average pressure coefficient at $n$ equally spaced points as a function of roll attitude for 4° angle of attack ($\delta = 4°$).**

### Solution of Equation

Equation (2) is of the general form

$$\sum_{j=l}^{n} \{a + b\sin[\varphi + 2\pi(j-1)/n]\}^2$$

with the conditions appearing as $\{a + b\sin[\varphi + 2\pi(j-1)/n]\} \geqq 0$, and $|a|, |b| \leqq 1$. These conditions restrict the possible values of $j$ for given values of $\varphi$, $n$, $a$, and $b$. To reduce the equation, one notices that, upon substitution of the exponential form of the sine function, the above expression becomes the sum of finite geometric series. Application of the formula for such series and some further algebraic manipulation allow the equation to be written as

$$\sum_{j=l}^{k} \{a + b\sin[\varphi + 2\pi(j-1)/n]\}^2 = (a^2 + b^2/2)(k - l + 1)$$
$$+ 2ab\sin[\varphi + \pi(k + l - 2)/n]\sin[\pi(k - l + 1)/n]/\sin(\pi/n)$$
$$-(b^2/2)\cos[2\varphi + 2\pi(k + l - 2)/n]\sin[2\pi(k - l + 1)/n]/\sin(2\pi/n) \quad (3)$$

provided $n \neq 1,2$. The apparent problem with $n = 1,2$ is resolved by taking the appropriate limits which then reproduce the expected products. The values of $k$, $l$ are determined in general by the values of $n$, $\varphi$ and by the restrictions on $a$, $b$.

For the cases of present interest, one has $a \geqq |b|$; i.e., the angle of attack must be less than or equal to the local body slope. This circumstance allows the above relation to be reduced to

$$\sum_{j=l}^{n} \{a + b\sin[\varphi + 2\pi(j-1)/n]\}^2 = n(a^2 + b^2/2)$$

Thus, when $n \geqq 3$, Eq. (2) becomes

$$\langle C_p \rangle = C_{p,o}[\cos^2\alpha\sin^2\delta + (1/2)\sin^2\alpha\cos^2\delta] \quad (4)$$

For values of $a \geqq |b|$, the values of $l$, $k$ may not necessarily be 1, $n$, respectively, since they may be restricted by choices of $\varphi$, $n$, $a$, and $b$.

### Discussion

The average pressure coefficients corresponding to the particular cases $n = 1$, which represents the pressure at a single point, and $n = 2$ are determined directly from Eq. (2). They are

$$C_p = C_{p,o}[\cos\alpha\sin\delta + \sin\alpha\cos\delta\sin\varphi]^2 \quad (5)$$

and

$$\langle C_p \rangle = C_{p,o}[\cos^2\alpha\sin^2\delta + \sin^2\alpha\cos^2\delta\sin^2\varphi] \quad (6)$$

respectively, and as shown in the preceding section, the average coefficient for all other cases, including $n = \infty$, is $\langle C_p \rangle = C_{p,o}[\cos^2\alpha\sin^2\delta + \frac{1}{2}\sin^2\alpha\cos^2\delta]$. This equation expresses the result that the average pressure around an inclined body of revolution may be determined by considering as few as three points. It also shows that for small body slope, the average pressure is relatively insensitive to the angle of attack, because the contribution due to angle of attack never exceeds one-half that due to body slope. Further, it may be seen that if $\alpha \neq 0$, the average pressure is somewhat less than the average of the pressures at the windward ($\varphi = \pi/2$) and leeward ($\varphi = -\pi/2$) meridians, where values are extremes, the difference of the two average coefficients being $\frac{1}{2}C_{p,o}\sin^2\alpha\cos^2\delta$. Finally, when the local body slope is small, the equation reduces to the approximation $\langle C_p \rangle \simeq \alpha^2 + 2\delta^2$, where the stagnation-pressure coefficient has been taken as the Newtonian value, 2.0. With that equation, the maximum errors in both the pressure and the pressure coefficient are no more than 4% for $\delta \leqq 10°$.

The effect of varying the number of orifices on a body has been calculated for an arbitrary configuration using Eqs. (4–6). The results are summarized in Figs. 1 and 2 where the average pressure coefficient, normalized by the stagnation-pressure coefficient, is shown as a function of the body roll position for several angles of attack. It may be observed that when $n = 1$, large pressure variations occur as a result of changes in roll position, and the variations increase with increasing angle of attack. When

$n = 2$, the sensitivity to roll position is markedly reduced; and, finally, for $n \geqq 3$, the pressure becomes independent of the roll position at all angles of attack.

## Conclusion

A closed-form equation is obtained which describes the average pressure acting at $n$ equally-spaced points located around the circumference of a body of revolution with an angle of attack in hypersonic flow. It is found that when $n \geqq 3$, the average normalized pressure depends only upon the local body slope and the total angle of attack, and is independent of both the aerodynamic roll angle and the number of points.

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# A Finite-Difference Method for Boundary Layers with Reverse Flow

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BOUNDARY LAYER separation occurs in virtually all aerodynamic flight regimes. Although interest in the details of this flow problem has existed for several decades, quantitative estimates based on numerical solutions of the governing equations have only recently been documented[1] where here the principal difficulty encountered is reported to be the presence of reverse flow. Since this, in effect, delivers information from a downstream point, it is identified with several numerical instabilities observed when one attempts to use a numerical finite-difference scheme to march into a separation bubble. Although this may certainly be true, it is also interesting to note that solutions of the self-similar form of the governing equations, which do not contain the convective terms believed to be causing numerical difficulties, also suffer numerical instabilities when finite-difference methods are applied to their solution. In particular, it is found that attempts to recover Stewartson's[2] reverse flow profiles for the Falkner-Skan[3] equation using finite differences fail, and only the forward profile solution is recovered. Only the shooting techniques[4,5] have thus far been able to overcome this deficiency. Unfortunately, shooting techniques have not yet been successfully carried over to nonsimilar boundary-layer calculations and, in fact, are found to be rather tedious to apply for the self-similar cases.

The present works demonstrates the application of a finite-difference technique that readily recovers Stewartson's reverse flow profiles without difficulty. The tedium of shooting methods is eliminated and the present method is shown to be reasonably fast and very insensitive to errors in initial data, overcoming up to a 30% error (purposely introduced) to recover the exact solution. It is anticipated that this method will be helpful in overcoming some of the difficulties presently being encountered in nonsimilar boundary-layer separation studies, and in any boundary-layer problem with nonunique solutions.

## Solution of the Governing Equations

The Falkner-Skan equations can be written in Görtler variables as two equations

$$F_{\eta\eta} - VF_\eta + \beta(1 - F^2) = 0 \tag{1a}$$

$$F + V_\eta = 0 \tag{1b}$$

where the pressure-gradient parameter is

$$\beta = (2\xi/u_e)(du_e/d\xi) \tag{2}$$

and the boundary conditions are

$$F(0) = V(0) = 0 \tag{3a}$$

and

$$F(\infty) = 1 \tag{3b}$$

Note that in this form, Eq. (1a) represents a transformed version of the momentum equation with $F$ a normalized longitudinal velocity, and Eq. (1b) gives the continuity equation, with $V$ related to the velocity normal to the surface. The variables monitored in the present study are the wall stress, given by

$$\tau_w = (\partial F/\partial\eta)_w \tag{4}$$

and a transformed displacement thickness, defined by

$$\delta \equiv \int_0^\infty (1 - F)d_\eta \tag{5}$$

In the preceding equations for $\beta \geqq 0$, accelerated flow and unique solutions result, while for $-0.19884 \leqq \beta \leqq 0$, a nonuniqueness is encountered. In this case, there are an infinite number of solutions of the governing equations, but only two are acceptable in that they have an exponential approach to the outer boundary conditions. These are the usual forward flow profiles of Falkner and Skan[3] and the reversed flow solutions, first identified by Stewartson.[2]

To recover these latter solutions, note first that the continuity equation and boundary conditions imply that

$$V \to -\eta + \delta \text{ as } \eta \to \infty \tag{6}$$

This result can be used to obtain numerical solutions to the continuity equation for any given value of $\delta$, since an integration of Eq. (1b) could proceed inward from the outer edge, $\eta_e$. However, it is found numerically that only two such solutions satisfy the boundary condition $V_w = V(0)$. This is demonstrated in Fig. 1, where the value of $V_w$ is given as a function of $\delta$ for $\beta = -0.18$. These latter solutions were obtained from fiinite-difference solutions of Eqs. (1–3) with $V(0) = 0$ replaced with Eq. (6) at some large $\eta = \eta_{edge}$.

As one might suspect the lower value of $\delta$ corresponds to the attached flow solution, whereas the reverse flow solution is given by the larger displacement thickness. The solution technique used here works in the $V_w$ vs $\delta$ plane and, simply stated, uses a reverse integration of the continuity equation to identify the largest value of $\delta$ which gives $V_w = 0$.

To solve this problem numerically, first denote the exact solutions of the governing equations as $F$ and $V$, and corresponding trial solutions as $F_G$ and $V_G$ the relations between these values being written as

$$V = V_G + v \tag{7a}$$

$$F = F_G + f \tag{7b}$$

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